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Volume-preserving maps, source-free systems and their local structures

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Abstract

In this paper, we study local structures of volume-preserving maps and source-free vector fields, which are defined in the Euclidean n-space $\mathbf{R}^n$ with $n \geqslant 3$. First, we prove that any volume-preserving map, defined in some neighbourhood of the origin, can be represented as a composition of $n-1$ essentially two-dimensional area-preserving maps. This result can be viewed as an analogue of the following known fact (Feng and Shang 1995 Volume-preserving algorithms for source-free dynamical systems *Numer. Math.* **71** 451–63): any source-free vector field on $\mathbf{R}^n$ can be represented as a sum of $n-1$ essentially two-dimensional Hamiltonian vector fields. Then, we present a local representation of source-free vector fields under volume-preserving coordinate changes. Finally, we construct a Lie algebra of skew-symmetric tensor potentials of second order associated with source-free vector fields. The Lie algebra turns out to be isomorphic to the Lie algebra of source-free vector fields.

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1. Introduction

In this paper, we first study volume-preserving maps locally defined in Euclidean n-space $\mathbf{R}^n$ with $n \geqslant 3$. The main result to be proved is that any volume-preserving map defined in some neighbourhood of the origin can be represented as a composition of $n-1$ essentially two-dimensional area-preserving maps. Here, an essentially two-dimensional area-preserving map is a volume-preserving map which leaves some $n-2$ coordinate functions invariant. This result can be naturally conjectured from its infinitesimal analogue which was observed in [1]: any source-free vector field on $\mathbf{R}^n$ can be represented as a sum of $n-1$ essentially two-dimensional Hamiltonian fields. This is a key observation leading to a nice construction of volume-preserving integrators for source-free systems [1]. Note that the Lie algebra of the Lie group of volume-preserving diffeomorphisms is just formed by source-free vector fields. The proof of the main result, however, does not follow directly from the Lie group and Lie algebra correspondence and, therefore, is necessary to be presented.

This result gives a new way to generate, in the local sense, general volume-preserving maps, because any two-dimensional area-preserving map can be completely specified by a so-called generating function. As a result, a volume-preserving mapping of n-dimensions is completely specified by $n-1$ functions—this is also a natural consequence of the volume-preserving property.

The problem of constructing general volume-preserving maps was first raised, to my knowledge, by Thyagaraja and Haas, who found a type of generating function to represent the most general volume-preserving mappings of three dimensions which are homotopic to the identity map [2]. This problem is relevant to solving source-free dynamical systems which are of great importance in many branches of physics. The generating function approach of Thyagaraja and Haas was systematically generalized by the author to arbitrary dimensions, with the development of Hamilton–Jacobi equations for source-free systems [3]. Quispel discovered independently a similar scheme to generate volume-preserving maps [4]. McLachlan and Quispel discussed the generating functions for dynamical systems in a very general setting [5]. These various types of generating functions have been applied to construct volume-preserving algorithms for numerically solving source-free systems [1, 2, 4, 6].

In [7], Sternberg proved a normal form theorem for C^∞ volume-preserving maps defined in neighbourhoods of the origin, which is assumed to be a fixed point of the maps. Sternberg's theorem generalizes the corresponding result of Moser in two dimensions [8]. It turns out that the set of germs of these normal forms falls into a finite number of classes, each of which is a maximal commutative subgroup of the group of such volume-preserving maps. We observe in this paper that each element of the maximal commutative subgroup is the composition of $n-1$ essentially two-dimensional area-preserving maps which are normal forms and commute with one another.

Source-free vector fields are the infinitesimal counterparts of volume-preserving diffeomorphisms. In section 3, we will study the representation of source-free vector fields under volume-preserving coordinate changes. In three dimensions, the representation has a very elegant form.

Source-free vector fields with the usual Jacobi–Lie bracket form a Lie algebra, which can be completely expressed by skew-symmetric tensor potentials of second order [9, 1]. The expression of source-free vector fields in terms of their tensor potentials automatically gives a linear homomorphism $\mathcal{X} : \mathbf{TP}_n \to \mathbf{SV}_n$, where $\mathbf{TP}_n$ is the space of all skew-symmetric tensor fields of second order and $\mathbf{SV}_n$ is the space of source-free vector fields on $\mathbf{R}^n$. In section 4, we introduce a bracket operation in $\mathbf{TP}_n$ so that the bracket, naturally induced from it by considering the module space $\mathbf{TP}_n/\mathbf{TP}_n^c$ instead of considering the space $\mathbf{TP}_n$ itself, is in fact a Lie bracket, where $\mathbf{TP}_n^c$ is the centre of $\mathbf{TP}_n$ under the former bracket. Moreover, the corresponding linear homomorphism induced from $\mathcal{X}$ is a Lie-algebra isomorphism between $\mathbf{TP}_n/\mathbf{TP}_n^c$ and $\mathbf{SV}_n$.

2. Representations of local volume-preserving diffeomorphisms with application to volume-preserving integrators

In this section, we first prove some representation results for volume-preserving local diffeomorphisms and then discuss the relationship of the representation with the volume-preserving integrators constructed in [1].

Theorem 1. *Let S be a C^∞ map from some neighbourhood of the origin of $\mathbf{R}^n$ into $\mathbf{R}^n$. Assume that S preserves the volume form*

$$\alpha = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \cdots \wedge \mathrm{d}x_n, \tag{2.1}$$

where $x_1, x_2, \ldots, x_n$ are Euclidean coordinates of $\mathbf{R}^n$. Then there exist $n-1$ essentially two-dimensional area-preserving C^∞ maps $S_1, \ldots, S_{n-1}$, which are well defined in a neighbourhood of the origin, such that $S = S_{n-1} \circ \cdots \circ S_1$. If S keeps the origin fixed, then we may determine S_j, $j = 1, 2, \ldots, n-1$, in such a way that they keep the origin fixed too.

Proof. Let $S : (x_1, x_2, \ldots, x_n) \to (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ be given by

$$\hat{x}_i = s_i(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{2.2}$$

and let

$$S(0) = \left(\hat{x}_1^{(0)}, \hat{x}_2^{(0)}, \ldots, \hat{x}_n^{(0)}\right).$$

Because of the volume preservation of S, we may assume

$$\left.\frac{\partial s_1}{\partial x_{i_1}}(x)\right|_{x=0} \neq 0 \tag{2.3}$$

for some $1 \leqslant i_1 \leqslant n$. By the implicit function theorem, we can solve x_{i_1} in terms of $\hat{x}_1$ from the first equation of (2.2), taking other coordinate variables involved in the equation as parameters. Then, we have

$$x_{i_1} = \hat{s}_1^{(i_1)}(w)$$

with $\hat{s}_1^{(i_1)}$ being a uniquely determined C^∞ function of variables w in a neighbourhood of w_0, where

$$w = \begin{cases} (\hat{x}_1, x_2, \ldots, x_n), & \text{if } i_1 = 1 \\ (x_1, \ldots, x_{i_1-1}, \hat{x}_1, x_{i_1+1}, \ldots, x_n), & \text{otherwise} \end{cases}$$

and

$$w_0 = \begin{cases} \left(\hat{x}_1^{(0)}, 0, \ldots, 0\right), & \text{if } i_1 = 1 \\ \left(0, \ldots, 0, \hat{x}_1^{(0)}, 0, \ldots, 0\right), & \text{otherwise.} \end{cases}$$

We define S_1, by distinguishing the cases $i_1 = 1$ and $i_1 \neq 1$, as follows:

(i) $i_1 = 1$. Define $S_1 : (x_1, x_2, \ldots, x_n) \to (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ by

$$\begin{cases} x_1 = \hat{s}_1^{(1)}(\hat{x}_1, x_2, \ldots, x_n) \\ \hat{x}_2 = \int_0^{x_2} \frac{\partial \hat{s}_1^{(1)}}{\partial \hat{x}_1}(\hat{x}_1, t, x_3, \ldots, x_n)\,\mathrm{d}t \\ \hat{x}_j = x_j, \qquad j = 3, \ldots, n; \end{cases} \tag{2.4}$$

(ii) $i_1 \neq 1$. Define $S_1 : (x_1, x_2, \ldots, x_n) \to (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ by

$$\begin{cases} x_{i_1} = \hat{s}_1^{(i_1)}(x_1, \ldots, x_{i_1-1}, \hat{x}_1, x_{i_1+1}, \ldots, x_n) \\ \hat{x}_{i_1} = \int_0^{x_1} \frac{\partial \hat{s}_1^{(i_1)}}{\partial \hat{x}_1}(t, \ldots, x_{i_1-1}, \hat{x}_1, x_{i_1+1}, \ldots, x_n)\,\mathrm{d}t \\ \hat{x}_j = x_j, \qquad j \neq 1, i_1. \end{cases} \tag{2.4$'$}$$

In each case, S_1 maps a neighbourhood of the origin to a neighbourhood of $\left(\hat{x}_1^{(0)}, 0, \ldots, 0\right)$ and is an essentially two-dimensional area-preserving mapping of C^∞ class. Therefore, $S^{(1)} = S \circ S_1^{-1}$ is volume preserving and has the following form:

$$\begin{cases} \hat{x}_1 = x_1 \\ \hat{x}_2 = s_2^{(1)}(x_1, x_2, \ldots, x_n) \\ \cdots \\ \hat{x}_n = s_n^{(1)}(x_1, x_2, \ldots, x_n) \end{cases} \tag{2.5}$$

which maps a neighbourhood of $S_1(0) = \left(\hat{x}_1^{(0)}, 0, \ldots, 0\right)$ to a neighbourhood of $S(0)$. This shows that $S^{(1)}$ is an essentially $(n-1)$-dimensional C^∞ volume-preserving mapping. In the case S keeps the origin fixed, the above constructed map S_1 and, therefore, the induced map $S^{(1)}$ keep the origin fixed too. If $n = 3$, then the theorem is already proved with $S_2 = S^{(1)}$. If $n > 3$, then the volume-preserving property of $S^{(1)}$ implies that

$$\frac{\partial s_2^{(1)}}{\partial x_{i_2}}(x)\bigg|_{x=0} \neq 0$$

for some $2 \leqslant i_2 \leqslant n$. The complete proof of the theorem is easily carried out by induction. □

The above decomposition preserves the linearity structure in the sense that each $S_i, i = 1, 2, \ldots, n-1$, is linear whenever S is a linear map. The decomposition also preserves the analyticity of the maps, i.e., if S is analytic, then each S_i is analytic too. But if S is of C^k with finite k, then $S_i, i = 1, 2, \ldots, n-1$, will be a map of C^{k-i} class according to the construction of the above decomposition. In this case, we need to assume $k \geqslant n$.

If S is a polynomial map, the above decomposition does not automatically imply that each S_i is also polynomial. An interesting problem is: whether a volume-preserving polynomial map can be written as a composition of some volume-preserving polynomial maps each of which is either linear or has a linear invariant function. If this is not the case, it is still interesting to study what kind of volume-preserving polynomial maps has such a decomposition. This is really a hard problem because it is closely related to the well-known Jacobian conjecture—a long standing unsolved problem: a volume-preserving polynomial map $f : \mathbf{C}^n \to \mathbf{C}^n$ is bijective and has a polynomial inverse. Lomeli and Meiss studied the normal forms of quadratic volume-preserving maps in three dimensions and only proved a very special result [10]: any quadratic volume-preserving map $f : \mathbf{R}^3 \to \mathbf{R}^3$, that has a quadratic inverse, can be written as the composition of an affine volume-preserving map T and a quadratic shear S, $f = T \circ S$, where $S(x) = x + \frac{1}{2}(x^T P x)v, v \in \mathbf{R}^3$ and P is a symmetric matrix such that $Pv = 0$. This result generalizes a result by Moser for quadratic symplectic maps [11]. Note that a quadratic shear not only has a linear invariant function but also has very simple dynamics.

The decomposition discussed above is closely related to the construction of volume-preserving integrators for source-free systems. It was proved in [1] that the map $S : (x_1, x_2, \ldots, x_n) \to (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, implicitly given by the following formula:

$$\begin{cases} \hat{x}_1 = x_1 + A_1(\hat{x}_1, x_2, \ldots, x_n), \\ \hat{x}_j = x_j + A_j(\hat{x}_1, \ldots, \hat{x}_j, x_{j+1}, \ldots, x_n) \\ \qquad + \int_{x_j}^{\hat{x}_j} \sum_{l=1}^{j-1} \frac{\partial A_l}{\partial x_l}(\hat{x}_1, \ldots, \hat{x}_{j-1}, t, x_{j+1}, \ldots, x_n)\, \mathrm{d}t, \qquad j = 2, \ldots, n-1, \\ \hat{x}_n = x_n + A_n(\hat{x}_1, \ldots, \hat{x}_{n-1}, x_n), \end{cases} \tag{2.6}$$

is volume preserving if the C^∞ vector field $A = (A_1, A_2, \ldots, A_n)^T$ is source free in the sense that

$$\mathrm{div}_\alpha A = \sum_{j=1}^{n} \frac{\partial A_j}{\partial x_j} = 0 \tag{2.7}$$

and if (2.6) does define a mapping. Here, div_α denotes the divergence operator with respect to the canonical volume form α. Equation (2.6) gives a volume-preserving integrator of first order if we take $A(x) = ha(x)$ with h denoting the time step size and $a(x)$ any source-free vector field. It was already known in [1] that any map defined from (2.6) by source-free vector field $A = (A_1, A_2, \ldots, A_n)^T$ is the composition of $n - 1$ essentially two-dimensional area-preserving maps. Theorem 1 shows that this decomposition property is shared in fact by the most general volume-preserving maps. Next, we show that (2.6) is also a formula to generate the most general near-identity volume-preserving maps by source-free vector fields $A = (A_1, A_2, \ldots, A_n)^T$.

Theorem 2. *For any C^∞ volume-preserving map $S : (x_1, \ldots, x_n) \to (\hat{x}_1, \ldots, \hat{x}_n)$ which keeps the origin fixed and has the identity linear part at the origin, there exists a source-free vector field $A = (A_1, A_2, \ldots, A_n)^T$, of C^∞ class, such that S is generated by A from (2.6).*

Proof. By assumption, we can write S in the form

$$\hat{x}_i = x_i + r_i(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{2.8}$$

with the functions r_i satisfying

$$|r_i(x)| \leqslant M_1 \|x\|^2, \qquad i = 1, 2, \ldots, n, \quad \text{if} \quad \|x\| \leqslant \delta_1$$

for some constants $M_1 > 0$ and $\delta_1 > 0$. Solving x_1 in terms of $\hat{x}_1$ from the first equation of (2.8), taking other variables as parameters, we obtain a uniquely determined C^∞ function A_1 of n variables in a neighbourhood of the origin such that

$$x_1 = \hat{x}_1 - A_1(\hat{x}_1, x_2, \ldots, x_n), \tag{2.9}$$

which is equivalent to the first equation of (2.8). Inserting (2.9) into the second equation of (2.8) and then solving x_2 in terms of $\hat{x}_2$ from the resulted equation, taking variables $\hat{x}_1, x_3, \ldots, x_n$ as parameters, we obtain

$$x_2 = \hat{x}_2 - \widetilde{r}_2(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_n)$$

with a uniquely determined C^∞ function $\widetilde{r}_2$ by the implicit function theorem. Let

$$A_2(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_n) = \widetilde{r}_2(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_n) - \int_{x_2}^{\hat{x}_2} \frac{\partial A_1}{\partial \hat{x}_1}(\hat{x}_1, t, x_3, \ldots, x_n)\,\mathrm{d}t \bigg|_{x_2 = \hat{x}_2 - \widetilde{r}_2(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_n)}.$$

Then the second equation of (2.8) is equivalent to the following:

$$\hat{x}_2 = x_2 + A_2(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_n) + \int_{x_2}^{\hat{x}_2} \frac{\partial A_1}{\partial \hat{x}_1}(\hat{x}_1, t, x_3, \ldots, x_n)\,\mathrm{d}t. \tag{2.10}$$

Now we use the induction argument to construct the vector field $A = (A_1, A_2, \ldots, A_n)$. For this, assume that, for $j = 1, 2, \ldots, k$, A_j have been defined well and the first k equations of (2.8) are equivalent to the following:

$$\hat{x}_j = x_j + A_j(\hat{x}_1, \ldots, \hat{x}_j, x_{j+1}, \ldots, x_n) + \int_{x_j}^{\hat{x}_j} \sum_{l=1}^{j-1} \frac{\partial A_l}{\partial x_l}(\hat{x}_1, \ldots, \hat{x}_{j-1}, t, x_{j+1}, \ldots, x_n)\,\mathrm{d}t, \qquad j = 1, 2, \ldots, k. \tag{2.11}$$

In $(2.11)_k$, the summation $\sum_{l=1}^{j-1}$ is understood to be zero when $j = 1$. To define A_{k+1}, we first solve $x_1, x_2, \ldots, x_k$ in terms of $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k$ from the equations of $(2.11)_k$, taking other variables as parameters, and then insert them into the $(k+1)$th equation of (2.8) and solve x_{k+1} in terms of $\hat{x}_{k+1}$ from the resulted equation, we obtain

$$x_{k+1} = \hat{x}_{k+1} - \widetilde{r}_{k+1}(\hat{x}_1, \ldots, \hat{x}_{k+1}, x_{k+2}, \ldots, x_n)$$

which is equivalent to the $(k+1)$th equation of (2.8), where $\widetilde{r}_{k+1}$ is a uniquely determined C^∞ function by the implicit function theorem. A_{k+1} is then defined as follows:

$$A_{k+1}(\hat{x}_1, \ldots, \hat{x}_{k+1}, x_{k+2} \ldots, x_n) = \widetilde{r}_{k+1}(\hat{x}_1, \ldots, \hat{x}_{k+1} x_{k+2}, x_n) - \int_{x_{k+1}}^{\hat{x}_{k+1}} \sum_{l=1}^{k} \frac{\partial A_l}{\partial \hat{x}_l}(\hat{x}_1, \ldots, \hat{x}_k, t, x_{k+2} \ldots, x_n)\, \mathrm{d}t \bigg|_{x_{k+1} = \hat{x}_{k+1} - \widetilde{r}_{k+1}(\hat{x}_1, \ldots, \hat{x}_{k+1}, x_{k+2}, \ldots, x_n)}. \tag{2.12}$$

Now we have already constructed the vector field $A = (A_1, A_2, \ldots, A_n)^T$, which is C^∞-smooth in some neighbourhood of the origin, and the map S is re-expressed by A from $(2.11)_n$. The proof of theorem 2 is closed by the following lemma. □

Lemma 1. *The vector field A constructed above is source free if the map S is volume preserving.*

Proof. With $A_1, \ldots, A_{n-1}$ constructed above, let

$$\widetilde{A}_n(x_1, \ldots, x_n) = -\int_0^{x_n} \sum_{l=1}^{n-1} \frac{\partial A_l}{\partial x_l}(x_1, \ldots, x_{n-1}, t)\, \mathrm{d}t.$$

Then the vector field $\widetilde{A} = (A_1, \ldots, A_{n-1}, \widetilde{A}_n)^T$ is source free, i.e.,

$$\frac{\partial \widetilde{A}_n}{\partial x_n} + \sum_{l=1}^{n-1} \frac{\partial A_l}{\partial x_l} = 0.$$

Define the map $\widetilde{S} : (x_1, x_2, \ldots, x_n) \to (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ by $A = (A_1, A_2, \ldots, A_n)^T$ from $(2.11)_n$ where A_n is replaced by $\widetilde{A}_n$. Then $\widetilde{S}$ is volume preserving [1]. It is clear that the map $E = S \circ \widetilde{S}^{-1} : (x_1, x_2, \ldots, x_n) \to (w_1, w_2, \ldots, w_n)$ is well defined in some neighbourhood of the origin and has the form

$$w_j = x_j, \qquad j = 1, \ldots, n-1, \qquad w_n = x_n + e_n(x_1, x_2, \ldots, x_n).$$

On the other hand, the volume-preserving property of S implies that E is volume preserving as well, which shows that e_n does not depend on x_n. It is easily verified, however, that

$$e_n(x) = A_n(x) - \widetilde{A}_n(E^{-1}(x)) + \int_{x_n - e_n(x)}^{x_n} \sum_{l=1}^{n-1} \frac{\partial A_l}{\partial x_l}(x_1, \ldots, x_{n-1}, t)\, \mathrm{d}t, \tag{2.13}$$

where $x = (x_1, x_2, \ldots, x_n)$. Taking derivative with respect to x_n on both sides of (2.13) with noting that $e_n(x)$ does not depend on x_n and that $\widetilde{A}$ is source free, we get

$$\sum_{l=1}^{n} \frac{\partial A_l}{\partial x_l}(x) = 0.$$

Lemma 1 is then proved. □

We denote by $S(A)$ the map from x to $\hat{x}$ determined by the vector field $A = (A_1, A_2, \ldots, A_n)^T$ from (2.6). One observes from [1], sections 3 and 5, that for source-free vector field $A = (A_1, A_2, \ldots, A_n)^T$, the associated map $S(A)$ has a decomposition of the form $S(A) = S(A^{(n-1)}) \circ \cdots \circ S(A^{(2)}) \circ S(A^{(1)})$, where $A = \sum_{k=1}^{n-1} A^{(k)}$ with

$$A^{(k)} = \left(0, \ldots, \frac{\partial b_{k,k+1}}{\partial x_{k+1}}, -\frac{\partial b_{k,k+1}}{\partial x_k}, 0, \ldots, 0\right)^T$$

and $b_{k,k+1}$ is given by A as follows (see [1]):

$$b_{12} = \int_0^{x_2} A_1\,\mathrm{d}x_2,$$

$$b_{k,k+1} = \int_0^{x_{k+1}} \left(A_k + \frac{\partial b_{k-1,k}}{\partial x_{k-1}}\right)\mathrm{d}x_{k+1}, \qquad 2 \leqslant k \leqslant n-2,$$

$$b_{n-1,n} = \int_0^{x_n} \left(A_{n-1} + \frac{\partial b_{n-2,n-1}}{\partial x_{n-2}}\right)\mathrm{d}x_n - \int_0^{x_{n-1}} A_n\bigg|_{x_n=0}\mathrm{d}x_{n-1}.$$

Note that here $A^{(k)}$ are essentially two-dimensional Hamiltonian vector fields and $S(A^{(k)})$ are defined in fact by the symplectic Euler method

$$\begin{cases} \hat{x}_j = x_j, & j \neq k, k+1 \\ \hat{x}_k = x_k + \frac{\partial b_{k,k+1}}{\partial x_{k+1}}(x_1, \ldots, x_{k-1}, \hat{x}_k, x_{k+1}, \ldots, x_n) \\ \hat{x}_{k+1} = x_{k+1} - \frac{\partial b_{k,k+1}}{\partial x_k}(x_1, \ldots, x_{k-1}, \hat{x}_k, x_{k+1}, \ldots, x_n) \end{cases} \tag{2.14}$$

which are essentially two-dimensional area-preserving maps. It is easily checked that in this way theorem 2 gives the same decomposition of a volume-preserving map as that described in the proof of theorem 1 where we take $i_1 = 1$ and $i_2 = 2$, etc. The interesting point is that there exists an obvious algebra-group correspondence between source-free vector fields A and the associated volume-preserving maps $S(A)$ with the remarkable distributive law:

$$\begin{array}{ccccccccc} A & = & A^{(n-1)} & + & \cdots & + & A^{(2)} & + & A^{(1)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S(A) & = & S(A^{(n-1)}) & \circ & \cdots & \circ & S(A^{(2)}) & \circ & S(A^{(1)}). \end{array}$$

Note that the Lie-algebra operation '+' satisfies the commutative law but the Lie-group operation '$\circ$' does not—the ordering of $S(A^{(k)})$, $k = 1, 2, \ldots, n-1$, in the composition formula above is uniquely determined by S. Therefore, the distributive law

$$S(A^{(n-1)} + \cdots + A^{(2)} + A^{(1)}) = S(A^{(n-1)}) \circ \cdots \circ S(A^{(2)}) \circ S(A^{(1)}) \tag{2.15}$$

is in fact an ordered distributive law. The correspondence S may be specified by the permutation $12\cdots(n-1)$, and we denote it by $S_{12\cdots(n-1)}$. For other permutations $\{i_1 i_2 \cdots i_{n-1}\}$ of $1, 2, \ldots, n-1$, one may get other correspondences $S_{i_1 i_2 \cdots i_{n-1}}$ with the corresponding ordered distributive law (2.15) where $S = S_{i_1 i_2 \cdots i_{n-1}}$.

Another point which should be remarked here is that we only consider a special type of normalizing conditions for uniqueness of the determination of a tensor potential $b = (b_{ij})$ of source-free vector field A (see [1]). For other normalizing conditions, one may get other decompositions of A and, accordingly, may get other correspondences.

Any other kind of symplectic methods for Hamiltonian systems, instead of symplectic Euler (2.14), may be applied to generate volume-preserving integrators for source-free systems. Therefore, one may also have other different kinds of decompositions and other different kinds of correspondences based on different kinds of essentially two-dimensional symplectic maps.

Sternberg proved in [7] a normal form reduction theorem of C^∞ volume-preserving maps near fixed points. The theorem says that any C^∞ volume-preserving transformation defined in some neighbourhood of the origin, keeping the origin fixed, can be brought by a volume-preserving change of coordinates to a normal form of the following form:

$$\hat{x}_i = x_i f_i(x_1 x_2 \cdots x_n), \qquad i = 1, 2, \ldots, n, \tag{2.16}$$

if the Jacobian matrix of the transformation at the origin satisfies a so-called regularity condition, where f_i are C^∞ functions of one variable with

$$\prod_{i=1}^{n} f_i = 1. \tag{2.17}$$

Sternberg also noted that the set of germs of these normal forms falls into a finite number of classes, each of which constitutes a maximal commutative subgroup of the group of local C^∞ volume-preserving maps. We remark here that any map of the normal form (2.16) with $f = (f_1, f_2, \ldots, f_n)$ satisfying (2.17) is a composition of the following $n-1$ essentially two-dimensional area-preserving maps:

$$\begin{cases} \hat{x}_k = x_k f_1 \cdots f_k \\ \hat{x}_{k+1} = x_{k+1} \dfrac{1}{f_1 \cdots f_k}, \qquad k = 1, 2, \ldots, n-1 \\ \hat{x}_j = x_j, \qquad j \neq k, k+1. \end{cases} \tag{2.18}$$

These $n-1$ essentially two-dimensional area-preserving maps are also in the class of normal forms and therefore commute with one another. This decomposition of volume-preserving normal forms is consistent with the decomposition for general volume-preserving maps described before.

The above decomposition results can be generalized to the case when the volume form is not canonical and, therefore, can be generalized to general Riemannian manifolds by using the arguments of McLachlan and Quispel in [5].

3. Source-free vector fields under volume-preserving coordinates transformations in dimension 3

The infinitesimal counterparts of volume-preserving diffeomorphisms are source-free vector fields, which constitute one of the simple infinite-dimensional Lie algebras of vector fields according to Cartan [12]. It may happen that the divergence of a vector field changes under coordinate transformations. Only volume-preserving coordinate transformations preserve the zero divergence of source-free vector fields. In this section, we give an invariant characterization of source-free vector fields under volume-preserving coordinate transformations. This characterization reveals some intrinsic property of source-free systems and is hoped will provide a new way to construct volume-preserving integrators.

A source-free vector field $X = (X_1, X_2, \ldots, X_n)^T$ on $\mathbf{R}^n$ can be expressed, at least locally, by an anti-symmetric tensor potential, say $a = (a_{ij})_{1 \leqslant i,j \leqslant n}$, of order 2 as follows:

$$X_i = \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}, \qquad i = 1, 2, \ldots, n. \tag{3.1}$$

The tensor potential a may be chosen, for uniqueness, to satisfy the following normalizing conditions:

$$a_{ij} = 0 \qquad \text{for} \quad |i-j| \geqslant 2 \tag{3.2}$$

as was done in [1]. So the dynamical system (phase flow) associated with the source-free vector field X is defined, at least locally, by the differential equations

$$\dot{x}_i = -\frac{\partial a_{i-1,i}}{\partial x_{i-1}} + \frac{\partial a_{i,i+1}}{\partial x_{i+1}}, \qquad i = 1, 2, \ldots, n \tag{3.3}$$

where we take $a_{01} = a_{n,n+1} = 0$, or more compactly,

$$\dot{x} = \sum_{k=1}^{n-1} J^{(k)} \nabla a_{k,k+1}. \tag{3.3$'$}$$

In (3.3) and (3.3′), we have used the notation $\dot{x}_i = \frac{\mathrm{d}x_i}{\mathrm{d}t}$, $\nabla h = \left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \ldots, \frac{\partial h}{\partial x_n}\right)^T$, the gradient of differentiable function $h : \mathbf{R}^n \to \mathbf{R}$ and

$$J^{(k)} = (J_{ij})_{1\leqslant i,j\leqslant n} \qquad \text{with} \qquad J_{ij} = -J_{ji} = \begin{cases} 1, & i = k,\ j = k+1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Below we examine the transformation properties of source-free systems under volume-preserving changes of coordinates. For this we start with the system of the form (3.3′) and assume a C^∞ volume-preserving change of coordinates $\Xi : \xi = (\xi_1, \xi_2, \ldots, \xi_n)^T \to x = (x_1, x_2, \ldots, x_n)^T$

$$x_i = \Xi_i(\xi_1, \xi_2, \ldots, \xi_n), \qquad i = 1, 2, \ldots, n. \tag{3.5}$$

Under transformation (3.5), system (3.3) turns into

$$\dot{\xi} = \sum_{k=1}^{n-1} \Xi_*^{-1}(\xi) J^{(k)} \Xi_*^{-T}(\xi) \nabla \widetilde{a}_{k,k+1}(\xi), \tag{3.6}$$

where $\Xi_*^{-1}(\xi)$ denotes the inverse of the Jacobian matrix of transformation (3.5) which is valued at ξ, $\Xi_*^{-T}(\xi)$ is the transpose of $\Xi_*^{-1}(\xi)$ and

$$\widetilde{a}_{k,k+1}(\xi) = a_{k,k+1}(\Xi(\xi)). \tag{3.7}$$

Ξ is volume preserving by assumption, we have $\det \Xi(\xi) = 1$ everywhere. Therefore,

$$\Xi_*^{-1}(\xi) = (\Xi_{ij}^*(\xi))_{1\leqslant i,j\leqslant n}, \qquad \text{with} \qquad \Xi_{ij}^*(\xi) = (-1)^{i+j} \det \Xi_*^{ij}(\xi), \tag{3.8}$$

where $\Xi_*^{ij}(\xi)$ is the $(n-1) \times (n-1)$ matrix obtained from $n \times n$ matrix $\Xi_*(\xi)$ by removing the ith row and jth column. In general, it is difficult to simplify equation (3.6) further if n is large. For $n = 3$, however, this equation may be reduced to a much simpler form.

Theorem 3. *For $n = 3$, equation (3.6) is reduced to*

$$\begin{cases} \dot{\xi}_1 = -\{\Xi_1, \widetilde{a}_{23}\}_{23} - \{\Xi_3, \widetilde{a}_{12}\}_{23}, \\ \dot{\xi}_2 = -\{\Xi_1, \widetilde{a}_{23}\}_{31} - \{\Xi_3, \widetilde{a}_{12}\}_{31}, \\ \dot{\xi}_3 = -\{\Xi_1, \widetilde{a}_{23}\}_{12} - \{\Xi_3, \widetilde{a}_{12}\}_{12} \end{cases} \tag{3.9}$$

if the coordinate change (3.5) is volume preserving. In (3.9), we used the notation

$$\{f, g\}_{ij} = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j} - \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}, \qquad 1 \leqslant i, j \leqslant 3 \tag{3.10}$$

for differentiable functions f and g of variables $\xi = (\xi_1, \xi_2, \xi_3)^T$.

Proof. It is clear that the matrix $\Xi_*^{-1} J^{(1)} \Xi_*^{-T}$ is skew-symmetric, therefore we may assume

$$\Xi_*^{-1} J^{(1)} \Xi_*^{-T} = \begin{pmatrix} 0 & c_{12} & c_{13} \\ -c_{12} & 0 & c_{23} \\ -c_{13} & -c_{23} & 0 \end{pmatrix}.$$

A simple calculation, with the use of the assumption that Ξ is volume preserving, gives

$$c_{12} = \begin{vmatrix} \frac{\partial \Xi_2}{\partial \xi_2} & \frac{\partial \Xi_2}{\partial \xi_3} \\ \frac{\partial \Xi_3}{\partial \xi_2} & \frac{\partial \Xi_3}{\partial \xi_3} \end{vmatrix} \begin{vmatrix} \frac{\partial \Xi_1}{\partial \xi_1} & \frac{\partial \Xi_1}{\partial \xi_3} \\ \frac{\partial \Xi_3}{\partial \xi_1} & \frac{\partial \Xi_3}{\partial \xi_3} \end{vmatrix} - \begin{vmatrix} \frac{\partial \Xi_1}{\partial \xi_3} & \frac{\partial \Xi_1}{\partial \xi_2} \\ \frac{\partial \Xi_3}{\partial \xi_3} & \frac{\partial \Xi_3}{\partial \xi_2} \end{vmatrix} \begin{vmatrix} \frac{\partial \Xi_2}{\partial \xi_3} & \frac{\partial \Xi_2}{\partial \xi_1} \\ \frac{\partial \Xi_3}{\partial \xi_3} & \frac{\partial \Xi_3}{\partial \xi_1} \end{vmatrix} = \frac{\partial \Xi_3}{\partial \xi_3},$$

$$c_{13} = \begin{vmatrix} \frac{\partial \Xi_2}{\partial \xi_2} & \frac{\partial \Xi_2}{\partial \xi_3} \\ \frac{\partial \Xi_3}{\partial \xi_2} & \frac{\partial \Xi_3}{\partial \xi_3} \end{vmatrix} \begin{vmatrix} \frac{\partial \Xi_1}{\partial \xi_2} & \frac{\partial \Xi_1}{\partial \xi_1} \\ \frac{\partial \Xi_3}{\partial \xi_2} & \frac{\partial \Xi_3}{\partial \xi_1} \end{vmatrix} - \begin{vmatrix} \frac{\partial \Xi_1}{\partial \xi_3} & \frac{\partial \Xi_1}{\partial \xi_2} \\ \frac{\partial \Xi_3}{\partial \xi_3} & \frac{\partial \Xi_3}{\partial \xi_2} \end{vmatrix} \begin{vmatrix} \frac{\partial \Xi_2}{\partial \xi_1} & \frac{\partial \Xi_2}{\partial \xi_2} \\ \frac{\partial \Xi_3}{\partial \xi_1} & \frac{\partial \Xi_3}{\partial \xi_2} \end{vmatrix} = -\frac{\partial \Xi_3}{\partial \xi_2},$$

$$c_{23} = \begin{vmatrix} \frac{\partial \Xi_2}{\partial \xi_3} & \frac{\partial \Xi_2}{\partial \xi_1} \\ \frac{\partial \Xi_3}{\partial \xi_3} & \frac{\partial \Xi_3}{\partial \xi_1} \end{vmatrix} \begin{vmatrix} \frac{\partial \Xi_1}{\partial \xi_2} & \frac{\partial \Xi_1}{\partial \xi_1} \\ \frac{\partial \Xi_3}{\partial \xi_2} & \frac{\partial \Xi_3}{\partial \xi_1} \end{vmatrix} - \begin{vmatrix} \frac{\partial \Xi_1}{\partial \xi_1} & \frac{\partial \Xi_1}{\partial \xi_3} \\ \frac{\partial \Xi_3}{\partial \xi_1} & \frac{\partial \Xi_3}{\partial \xi_3} \end{vmatrix} \begin{vmatrix} \frac{\partial \Xi_2}{\partial \xi_1} & \frac{\partial \Xi_2}{\partial \xi_2} \\ \frac{\partial \Xi_3}{\partial \xi_1} & \frac{\partial \Xi_3}{\partial \xi_1} \end{vmatrix} = \frac{\partial \Xi_3}{\partial \xi_1}.$$

Similarly,

$$\Xi_*^{-1} J^{(2)} \Xi_*^{-T} = \begin{pmatrix} 0 & \frac{\partial \Xi_1}{\partial \xi_3} & -\frac{\partial \Xi_1}{\partial \xi_2} \\ -\frac{\partial \Xi_1}{\partial \xi_3} & 0 & \frac{\partial \Xi_1}{\partial \xi_1} \\ \frac{\partial \Xi_1}{\partial \xi_2} & -\frac{\partial \Xi_1}{\partial \xi_1} & 0 \end{pmatrix}.$$

As a consequence, (3.9) is derived from (3.6) with the notation (3.10).

If a source-free system takes, instead of (3.3), the more general form

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}, \qquad i = 1, 2, \ldots, n, \tag{3.11}$$

with skew-symmetric tensor potentials $a = (a_{ij})_{1\leqslant i,j\leqslant n}$. Then for $n = 3$, under a volume-preserving coordinate transformation (3.5), (3.11) turns into the following form:

$$\begin{cases} \dot{\xi}_1 = -\{\Xi_1, \widetilde{a}_{23}\}_{23} + \{\Xi_2, \widetilde{a}_{13}\}_{23} - \{\Xi_3, \widetilde{a}_{12}\}_{23}, \\ \dot{\xi}_2 = -\{\Xi_1, \widetilde{a}_{23}\}_{31} + \{\Xi_2, \widetilde{a}_{13}\}_{31} - \{\Xi_3, \widetilde{a}_{12}\}_{31}, \\ \dot{\xi}_3 = -\{\Xi_1, \widetilde{a}_{23}\}_{12} + \{\Xi_2, \widetilde{a}_{13}\}_{12} - \{\Xi_3, \widetilde{a}_{12}\}_{12}, \end{cases} \tag{3.12}$$

and (3.9) is just the case of (3.12) where $a_{13} = 0$, which makes (3.11) become (3.3) for $n = 3$.

If coordinate transformation (3.5) is the identity, then (3.12) is nothing but (3.11) in the case $n = 3$. Therefore, (3.12) gives an alternative general form of source-free systems in three dimensions.

The generalization of the transformation formula of form (3.12) to higher dimensional cases does not seem possible, because for $n > 3$, the number of the components of a skew-symmetric tensor potential $a = (a_{ij})_{1\leqslant i,j\leqslant n}$ is strictly bigger than the number of the components of a coordinate transformation. □

Theorem 3 gives an invariant representation of source-free vector fields under volume-preserving coordinate transformation and, therefore, reveals an intrinsic property of source-free systems. We hope this characterization can help provide a new way to construct volume-preserving integrators for source-free systems. We would like to discuss this topic separately and do not give more developments in this paper.

4. Lie algebra of skew-symmetric tensor potentials

Skew-symmetric tensor potentials of second order were introduced to represent general source-free systems in a local sense. They were also successfully applied to construct volume-preserving integrators [1]. Theorem 3 gives a kind of invariant representation of source-free vector fields in dimension 3 under volume-preserving coordinate transformations by skew-symmetric tensor potentials. This representation has a very elegant and symmetric form, which contrasts with that of Hamiltonian vector fields under symplectic coordinate transformations on the symplectic space. In this section, we study the Lie-algebraic structure of the vector space of skew-symmetric tensor potentials, which should be naturally induced from the Lie algebra of source-free vector fields. This may also be regarded as a generalization of the scalar potential Lie algebra of Hamiltonian vector fields to tensor potential Lie algebra of source-free fields.

Let $\mathbf{V}_n$ be the Lie algebra of C^∞ smooth vector fields on $\mathbf{R}^n$ with the usual Jacobi–Lie bracket

$$[X, Y] = X_* Y - Y_* X, \tag{4.1}$$

where X_* denotes the Jacobian matrix of the vector field X. We consider the Lie subalgebra $\mathbf{SV}_n$ of source-free vector fields on $\mathbf{R}^n$. $\mathbf{SV}_n$ is a simple Lie algebra of infinite dimensions [12]. To any source-free vector field X, there corresponds a skew-symmetric tensor potential $a = (a_{ij})_{1\leqslant i,j\leqslant n}$ such that X is determined by a from (3.1). It is seen from section 3 that a is not uniquely determined in this way. For uniqueness, one has to require additional normalizing conditions.

Let $\mathbf{TP}_n$ be the set of all C^∞ skew-symmetric tensor potentials of form $a = (a_{ij})_{1\leqslant i,j\leqslant n}$. We would like to introduce a Lie algebraic structure on $\mathbf{TP}_n$ so that the linear map from $\mathbf{TP}_n$ to $\mathbf{SV}_n$ given by (3.1) is a Lie algebra homomorphism. If this were the case, then any two skew-symmetric tensor potentials $a = (a_{ij})_{1\leqslant i,j\leqslant n}$ and $b = (b_{ij})_{1\leqslant i,j\leqslant n}$ might associate with another skew-symmetric tensor potential $\{a, b\}$ such that

$$[X_a, X_b] = X_{\{a,b\}}, \tag{4.2}$$

where X_a denotes the source-free vector field with tensor potential a, i.e.,

$$X_a^i = \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}, \qquad i = 1, 2, \ldots, n \tag{4.3}$$

in components.

By (4.1), the ith component of the vector field $[X_a, X_b]$ equals to

$$[X_a, X_b]_i = \sum_{k=1}^n \left(\frac{\partial X_a^i}{\partial x_k} X_b^k - \frac{\partial X_b^i}{\partial x_k} X_a^k \right).$$

The use of free divergence of X_a and X_b leads to

$$[X_a, X_b]_i = \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(X_a^i X_b^k - X_a^k X_b^i \right),$$

which suggests the following possible definition of $\{a, b\}$:

$$\{a, b\}_{ij} = X_a^i X_b^j - X_a^j X_b^i, \qquad i, j = 1, 2, \ldots, n, \tag{4.4a}$$

or, in compact form,

$$\{a, b\} = X_a X_b^T - X_b X_a^T. \tag{4.4b}$$

The bracket defined above is skew-symmetric and bilinear. However, it does not satisfy the Jacobian identity and, therefore, is not a Lie bracket. We separate the cases $n = 3$ and $n > 3$.

4.1. Case $n = 3$

By direct but a little bit cumbersome calculations one easily verifies the identity

$$\{\{a, b\}, c\}_{i,j} + \{\{b, c\}, a\}_{ij} + \{\{c, a\}, b\}_{ij} = \frac{\partial H_{abc}}{\partial x_k} \tag{4.5}$$

for an even permutation (ijk) of (123), where

$$\begin{aligned} H_{abc} &= X_a^1\left(X_b^3 X_c^2 - X_b^2 X_c^3\right) + X_a^2\left(X_b^1 X_c^3 - X_b^3 X_c^1\right) + X_a^3\left(X_b^2 X_c^1 - X_b^1 X_c^2\right) \\ &= -(X_a, X_b \times X_c). \end{aligned} \tag{4.6}$$

Here, we denote by (X, Y) the inner product and by $X \times Y$ the vector product of three-dimensional vectors X and Y.

Equation (4.5) does not vanish in general, therefore, the above defined bracket is not a Lie bracket. Moreover, it seems difficult to modify the definition (4.4) so that $\{a, b\}$ is a Lie bracket and the relation (4.2) can be preserved.

It is observed, however, that the right-hand side of (4.5) has a special form. Define operator $\widetilde{\nabla} : C^\infty(\mathbf{R}^3) \to \mathbf{TP}_3$ by

$$\widetilde{\nabla} H = \begin{pmatrix} 0 & \frac{\partial H}{\partial x_3} & -\frac{\partial H}{\partial x_2} \\ -\frac{\partial H}{\partial x_3} & 0 & \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} & -\frac{\partial H}{\partial x_1} & 0 \end{pmatrix}, \tag{4.7}$$

then (4.5) becomes

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = \widetilde{\nabla} H_{abc} \tag{4.5$'$}$$

with H_{abc} given by (4.6). Let

$$\mathbf{TP}_3^c = \widetilde{\nabla}(C^\infty(\mathbf{R}^3)),$$

with the range of the operator $\widetilde{\nabla}$, we have

Lemma 2. *For $a \in \mathbf{TP}_3$, the following statements are equivalent:*

(i) $X_a = 0$*;*
(ii) $a \in \mathbf{TP}_3^c$*;*
(iii) $\{a, b\} = 0$ *for any* $b \in \mathbf{TP}_3$.

Let $\mathcal{X} : \mathbf{TP}_n \to \mathbf{SV}_n$ be the linear map defined by (4.3), then it is an onto (but not one to one) map. From lemma 2, it follows that $\mathbf{TP}_3^c$ is just the kernel of $\mathcal{X}$ in the case $n = 3$ (lemma 2, (i) and (ii)) and also the centre[1] of $\mathbf{TP}_3$ under the bracket $\{\cdot, \cdot\}$ (lemma 2, (ii) and (iii)). If we define the following equivalence relation in $\mathbf{TP}_3$:

$$a \sim b \qquad \text{if} \quad a - b \in \mathbf{TP}_3^c, \tag{4.8}$$

then the map, naturally induced by this equivalence,

$$\begin{aligned} \widetilde{\mathcal{X}} = \mathcal{X}/\sim &: \mathbf{TP}_3/\mathbf{TP}_3^c \to \mathbf{SV}_3 \\ &[a] \to X_{[a]} = X_a \end{aligned} \tag{4.9}$$

is a Lie algebra isomorphism, where $[a]$ denotes the class of elements equivalent to a. The Lie bracket of $\mathbf{TP}_3 \big/ \mathbf{TP}_3^c$, induced by this equivalence, is given as follows:

$$\{[a], [b]\} = [\{a, b\}]. \tag{4.10}$$

To conclude, we have

[1] The set of all $a \in \mathbf{TP}_n$ such that $\{a, b\} = 0$ for any $b \in \mathbf{TP}_n$ is called the centre of $\mathbf{TP}_n$. The centre is a linear subspace of the linear space $\mathbf{TP}_n$.

Theorem 4. *The bracket defined by (4.10) is a Lie bracket, therefore,* $\mathbf{TP}_3\big/\mathbf{TP}_3^c$*, with this Lie bracket, is a Lie algebra. Moreover, the map* $\widetilde{\mathcal{X}} : [a] \to X_{[a]}$ *given by (4.9) and (4.3) is a Lie-algebra isomorphism from* $\mathbf{TP}_3\big/\mathbf{TP}_3^c$ *to* $\mathbf{SV}_3$*, i.e., the following relation holds:*

$$[X_{[a]}, X_{[b]}] = X_{\{[a],[b]\}}, \tag{4.11}$$

which is nothing but (4.2).

4.2. Case $n > 3$

With the bracket defined by (4.4), one can verify the identities

$$\{\{a, b\}, c\}_{ij} + \{\{b, c\}, a\}_{ij} + \{\{c, a\}, b\}_{ij} = -\sum_{k=1}^{n} \frac{\partial H_{ij}^k(abc)}{\partial x_k} \qquad i, j = 1, 2, \ldots, n, \tag{4.12}$$

where

$$H_{ij}^k(abc) = X_a^i\big(X_b^j X_c^k - X_b^k X_c^j\big) + X_a^j\big(X_b^k X_c^i - X_b^i X_c^k\big) + X_a^k\big(X_b^i X_c^j - X_b^j X_c^i\big). \tag{4.13}$$

Lemma 3. *For any* $a, b, c \in \mathbf{TP}_n$, $\big(H_{ij}^k(abc)\big)_{1\leqslant i,j,k\leqslant n}$ *is a skew-symmetric tensor field of order 3 and is invariant with respect to even permutations of* (abc).

Equation (4.12) does not vanish in general and, therefore, the bracket defined by (4.4) is not a Lie bracket. Let $\mathbf{CTP}_n$ be the set of all C^∞ skew-symmetric tensor fields of order 3 on $\mathbf{R}^n$ and define the operator $\widetilde{\nabla} : \mathbf{CTP}_n \to \mathbf{TP}_n$ by

$$\widetilde{\nabla} H = a^H \tag{4.14}$$

with

$$a_{ij}^H = \sum_{k=1}^{n} \frac{\partial H_{ij}^k}{\partial x_k} \tag{4.15}$$

for $H = \big(H_{ij}^k\big)_{1\leqslant i,j,k\leqslant n} \in \mathbf{CTP}_n$, then (4.12) implies that

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} \in \widetilde{\nabla}(\mathbf{CTP}_n) \tag{4.16}$$

for any $a, b, c \in \mathbf{TP}_n$. Denote by $\mathbf{TP}_n^c$ $\widetilde{\nabla}(\mathbf{CTP}_n)$. The following lemma can easily be proved.

Lemma 4. *For* $a \in \mathbf{TP}_n$*, the following statements are equivalent:*

(i) $X_a = 0$;
(ii) $a \in \mathbf{TP}_n^c$;
(iii) $\{a, b\} = 0$ *for any* $b \in \mathbf{TP}_n$.

It follows from lemma 4 that $\mathbf{TP}_n^c$ is the kernel of the linear map $\mathcal{X} : \mathbf{TP}_n \to \mathbf{SV}_n$ (lemma 4, (i) and (ii)) and is the centre of $\mathbf{TP}_n$ under the bracket $\{\cdot, \cdot\}$ (lemma 4, (ii) and (iii)). By virtue of this fact, we may introduce the equivalence relation in $\mathbf{TP}_n$ as follows:

$$a \sim b \qquad \text{if} \quad a - b \in \mathbf{TP}_n^c,$$

and have the natural induced linear map

$$\widetilde{\mathcal{X}} = \mathcal{X}/\sim : \mathbf{TP}_n/\mathbf{TP}_n^c \to \mathbf{SV}_n$$
$$[a] \to X_{[a]} = X_a,$$

where $[a]$ is the class of elements equivalent to a. It is clear that the bracket defined by (4.10) is also well defined and a Lie bracket in the case $n > 3$ and the map $\widetilde{\mathcal{X}}$ is a Lie-algebra isomorphism from $\mathbf{TP}_n/\mathbf{TP}_n^c$ to $\mathbf{SV}_n$. To summarize, the parallel statements of theorem 4 are valid in the case $n > 3$.

As the referees mentioned, the exposition and proof of some results in sections 3 and 4 could be simpler and clearer using the concepts and notation of exterior differential forms. Source-free vector fields in $\mathbf{SV}_n$ are in 1–1 correspondence with closed $(n-1)$-forms in Λ^{n-1}, which can be characterized locally by the derivative (action of d_{n-2}) of $(n-2)$-forms, giving the skew-symmetric tensor potentials. The exact sequence

$$\cdots \to \Lambda^{n-3} \to \Lambda^{n-2} \to \Lambda^{n-1} \to \Lambda^{n} \to 0$$

of the exterior differential forms under the exterior derivatives gives $\Lambda^{n-2}/\mathrm{Range}(d_{n-3}) = \Lambda^{n-2}/\mathrm{Kernel}(d_{n-2})$, which is nothing but isomorphic to the module space $\mathbf{TP}_n\big/\mathbf{TP}_n^c$, as was characterized above. A general local representation of volume-preserving systems on a symplectic manifold was given in [13] where, more interestingly, the so-called Euler–Lagrange cohomological groups were also developed for source-free vector fields.

In the current paper, we did not adopt this more modern and geometrical framework. We adopted the coordinate formalism instead, because the coordinate language can help give an explicit invariant representation of source-free vector fields under volume-preserving coordinate transformations (section 3) and also help give an explicit characterization of the quotient elements by the Jacobian cyclic sum of any three tensor potentials (section 4). These results do not seem obvious even in the geometrical language. However, the exterior differential calculus and the corresponding geometrical arguments can really help one have a better understanding of the structures of volume-preserving systems and even more general systems of Lie type. Sections 3 and 4 of this paper result from the attempt to generalize the well-known Poisson representation of Hamiltonian systems to the volume-preserving case. For the Poisson structure and relevant geometrical theory, see [14–16].

The exterior differential calculus as well as topological or geometrical consideration has substantially come into numerical analysis in recent years (see [17–20] and references therein). Many structure-preserving finite elements have been constructed for elastic problems [18, 19] and for electromagnetic problems [17, 20]. From these examples of excellent works, one can see that the preservation of relevant topological structures of continuous systems in discretizations can naturally lead to remarkable numerical stability and give optimal error estimates in some sense. Discrete analogues of exterior differential calculus were developed by many authors (see, e.g., [21–25]).

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